

# Quasi-Local Conservation Equations in General Relativity

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A set of exact quasi-local conservation equations is derived from the Einstein's equations using the first-order Kaluza-Klein formalism of general relativity in the (2,2)-splitting of 4-dimensional spacetime. These equations are interpreted as quasi-local energy, momentum, and angular momentum conservation equations. In the asymptotic region of asymptotically flat spacetimes, it is shown that the quasi-local energy and energy-flux integral reduce to the Bondi energy and energy-flux, respectively. In spherically symmetric spacetimes, the quasi-local energy becomes the Misner-Sharp energy. Moreover, on the event horizon of a general dynamical black hole, the quasi-local energy conservation equation coincides with the conservation equation studied by Thorne *et al.* We discuss the remaining quasi-local conservation equations briefly.

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## I. INTRODUCTION AND KINEMATICS

In general relativity there have been many attempts to obtain quasi-local conservation equations [1–6]. One of the motivations of these efforts is that the quasi-local conservation equations allow us to predict certain aspects of the future of a quasi-local region of a given spacetime without actually solving the Einstein's equations for that region. Recall that in the Newtonian theory, the conservation of momentum  $\sum \vec{p} = \text{constant}$  immediately follows from Newton's third law, which is no more than the consistency condition implementing Newton's second law. In general relativity, the consistency conditions for evolution are already incorporated into the Einstein's equations through the constraint equations, from which *global* conservation equations were found. The purpose of this letter is to show that, from the Einstein's equations, one can find conservation equations of a stronger form, namely, *quasi-local* conservation equations. These equations, interpreted as quasi-local energy, momentum, and angular momentum conservation equations, are *exact* and *unique* in the sense that they are obtained by integrating the Einstein's “constraint” equations over a compact two-surface [7].

Let us start from the following line element [8–15]

$$ds^2 = -2dudv - 2hdu^2 + e^\sigma \rho_{ab} (dy^a + A_+^a du + A_-^a dv) (dy^b + A_+^b du + A_-^b dv), \quad (1)$$

where  $+, -$  stands for  $u, v$ , respectively. The geometry (1) can be understood as follows. The hypersurface  $u = \text{constant}$  is an out-going null hypersurface, and the hypersurface  $v = \text{constant}$  is either timelike, null, or spacelike, depending on the sign of  $2h$ . The intersection of two hypersurfaces  $u, v = \text{constant}$  defines a spacelike compact two-surface  $N_2$ , on which we introduce the coordinates  $y^a (a = 2, 3)$ . The metric on  $N_2$  is written as

$$\phi_{ab} = e^\sigma \rho_{ab}, \quad (2)$$

where  $\rho_{ab}$  is the conformal two-metric satisfying the condition that

$$\det \rho_{ab} = 1, \quad (3)$$

and  $e^\sigma$  is the area element. Notice that  $v$  is the affine parameter of the null vector field

$$\frac{\partial}{\partial v} - A_-^a \frac{\partial}{\partial y^a}, \quad (4)$$

which rules the out-going null hypersurface  $u = \text{constant}$ . If we further assume that  $A_-^a = 0$ , then the metric (1) becomes identical to the Newman-Unti metric [14]. In this letter, however, we shall retain the  $A_-^a$  field, since its presence will make the  $N_2$ -diffeomorphism invariant *Yang-Mills* type gauge theory aspect of this Kaluza-Klein formalism transparent. Let us mention that the coordinates used in the above construction are not unique, which

means that there are residual symmetries that preserve the metric (1). These residual symmetries consist of the diffeomorphisms of  $N_2$ , the reparametrization of  $u$ , and the shift of the origin of the affine parameter  $v$  at each point of  $N_2$ , and were studied in detail in [15].

The spacetime integral  $I_0$  of the scalar curvature of the metric (1) can be written as

$$I_0 = \int du dv d^2y L_0 + \text{surface integral}, \quad (5)$$

where  $L_0$  is given by [6,11–13]

$$\begin{aligned} L_0 = & -\frac{1}{2}e^{2\sigma}\rho_{ab}F_{+-}^aF_{+-}^b + e^\sigma(D_+\sigma)(D_-\sigma) - \frac{1}{2}e^\sigma\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) + e^\sigma R_2 \\ & - 2e^\sigma(D_-h)(D_-\sigma) - he^\sigma(D_-\sigma)^2 + \frac{1}{2}he^\sigma\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}). \end{aligned} \quad (6)$$

Here  $R_2$  is the scalar curvature of  $N_2$ , and we defined the  $\text{diff}N_2$ -covariant derivatives as follows,

$$F_{+-}^a = \partial_+ A_-^a - \partial_- A_+^a - [A_+, A_-]_L^a, \quad (7)$$

$$D_\pm\sigma = \partial_\pm\sigma - [A_\pm, \sigma]_L, \quad (8)$$

$$D_\pm h = \partial_\pm h - [A_\pm, h]_L, \quad (9)$$

$$D_\pm\rho_{ab} = \partial_\pm\rho_{ab} - [A_\pm, \rho]_{Lab}, \quad (10)$$

where  $\partial_+ = \partial/\partial u$ ,  $\partial_- = \partial/\partial v$ ,  $\partial_a = \partial/\partial y^a$ , and  $[A_\pm, *]_L$  is the Lie derivative of  $*$  along the vector field  $A_\pm := A_\pm^a \partial_a$ .

In addition to the eight equations of motion that follow directly from the integral  $I_0$  by the variational principle, there are two supplementary equations associated with the partial gauge-fixing of the general metric to the metric (1). These equations, from which two quasi-local conservation equations are shown to follow, are obtained by varying the Einstein-Hilbert action before the partial gauge-fixing condition is introduced. Here we present them without derivation, and they are given by [13]

$$\begin{aligned} \text{(i)} \quad & -e^\sigma D_+^2\sigma - \frac{1}{2}e^\sigma(D_+\sigma)^2 - e^\sigma(D_-h)(D_+\sigma) + e^\sigma(D_+h)(D_-\sigma) \\ & + 2he^\sigma(D_-h)(D_-\sigma) + e^\sigma F_{+-}^a \partial_a h - \frac{1}{4}e^\sigma\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) + \partial_a(\rho^{ab}\partial_b h) \\ & + he^\sigma\left\{R_2 - (D_+\sigma)(D_-\sigma) + \frac{1}{2}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) + \frac{1}{2}e^\sigma\rho_{ab}F_{+-}^aF_{+-}^b\right\} \\ & + h^2e^\sigma\left\{(D_-\sigma)^2 - \frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\right\} = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} \text{(ii)} \quad & e^\sigma D_+ D_- \sigma + e^\sigma D_- D_+ \sigma + 2e^\sigma(D_+\sigma)(D_-\sigma) - 2e^\sigma(D_-h)(D_-\sigma) \\ & - \frac{1}{2}e^{2\sigma}\rho_{ab}F_{+-}^aF_{+-}^b - e^\sigma R_2 - he^\sigma\left\{(D_-\sigma)^2 - \frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\right\} = 0. \end{aligned} \quad (12)$$

## II. A SET OF QUASI-LOCAL CONSERVATION EQUATIONS

Notice that the equations (11) and (12) are *first-order* in  $D_-$  derivatives, so that they may be regarded as two “constraint” equations. Thus, in this formalism, the *natural* vector field that defines the evolution is  $\partial_-$ . Then the momenta  $\pi_I = \{\pi_h, \pi_\sigma, \pi_a, \pi^{ab}\}$  conjugate to the configuration variables  $q^I = \{h, \sigma, A_+^a, \rho_{ab}\}$  are defined as

$$\pi_I := \frac{\partial L_0}{\partial(\partial_- q^I)}, \quad (13)$$

and are given by

$$\pi_h = -2e^\sigma(D_-\sigma), \quad (14)$$

$$\pi_\sigma = -2e^\sigma(D_-h) - 2he^\sigma(D_-\sigma) + e^\sigma(D_+\sigma), \quad (15)$$

$$\pi_a = e^{2\sigma}\rho_{ab}F_{+-}^b, \quad (16)$$

$$\pi^{ab} = he^\sigma\rho^{ac}\rho^{bd}(D_-\rho_{cd}) - \frac{1}{2}e^\sigma\rho^{ac}\rho^{bd}(D_+\rho_{cd}). \quad (17)$$

Notice that  $\pi^{ab}$  is traceless

$$\rho_{ab}\pi^{ab} = 0 \quad (18)$$

due to the identities

$$\rho^{ab}D_{\pm}\rho_{ab} = 0. \quad (19)$$

The ‘‘Hamiltonian’’ function  $H_0$  defined as

$$H_0 := \pi_I \partial_- q^I - L_0 \quad (20)$$

is found to be

$$H_0 = H - A_-^a C_a + \text{surface terms}, \quad (21)$$

where  $H$  and  $C_a$  are given by

$$\begin{aligned} H = & -\frac{1}{2}e^{-\sigma}\pi_{\sigma}\pi_h + \frac{1}{4}he^{-\sigma}\pi_h^2 - \frac{1}{2}e^{-2\sigma}\rho^{ab}\pi_a\pi_b + \frac{1}{2h}e^{-\sigma}\rho_{ac}\rho_{bd}\pi^{ab}\pi^{cd} \\ & + \frac{1}{2}\pi_h(D_+\sigma) + \frac{1}{2h}\pi^{ab}(D_+\rho_{ab}) + \frac{1}{8h}e^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) - e^{\sigma}R_2, \end{aligned} \quad (22)$$

$$\begin{aligned} C_a = & \partial_+\pi_a - \partial_b(A_+^b\pi_a) - \pi_b\partial_a A_+^b - \pi_{\sigma}\partial_a\sigma + \partial_a\pi_{\sigma} - \pi_h\partial_a h - \pi^{bc}\partial_a\rho_{bc} \\ & + \partial_b(\pi^{bc}\rho_{ac}) + \partial_c(\pi^{bc}\rho_{ab}) - \partial_a(\pi^{bc}\rho_{bc}). \end{aligned} \quad (23)$$

In terms of these canonical variables  $\{\pi_I, q^I\}$ , the supplementary equations (11) and (12) can be written as

$$\begin{aligned} \text{(i')} \quad & \pi^{ab}D_+\rho_{ab} + \pi_{\sigma}D_+\sigma - hD_+\pi_h - \partial_+(h\pi_h + 2e^{\sigma}D_+\sigma) \\ & + \partial_a(h\pi_h A_+^a + 2A_+^a e^{\sigma}D_+\sigma + 2he^{-\sigma}\rho^{ab}\pi_b + 2\rho^{ab}\partial_b h) = 0, \end{aligned} \quad (24)$$

$$\text{(ii')} \quad H - \partial_+\pi_h + \partial_a(A_+^a\pi_h + e^{-\sigma}\rho^{ab}\pi_b) = 0. \quad (25)$$

Moreover, we have two more first-order equations

$$C_a = 0, \quad (26)$$

which follows trivially by varying  $H_0$  with respect to  $A_-^a$  (or by varying the Einstein-Hilbert action [10,12], which requires rather lengthy computations). The equations (24), (25), and (26) are the four Einstein’s ‘‘constraint’’ equations in the gauge (1). Notice that the equations (24) and (25) are *divergence*-type equations. If we contract the equation (26) by an arbitrary function  $\xi^a$  such that

$$\partial_{\pm}\xi^a = 0, \quad (27)$$

then the resulting equation is also a divergence-type equation,

$$\begin{aligned} & \pi^{ab}\mathcal{L}_{\xi}\rho_{ab} + \pi_{\sigma}\mathcal{L}_{\xi}\sigma + \pi_h\mathcal{L}_{\xi}h + \pi_a\mathcal{L}_{\xi}A_+^a - \partial_+(\xi^a\pi_a) \\ & + \partial_a(-\xi^a\pi_{\sigma} + 2\pi^{ab}\xi^c\rho_{bc} + A_+^a\xi^b\pi_b) = 0, \end{aligned} \quad (28)$$

where  $\mathcal{L}_{\xi}$  is the Lie derivative along  $\xi := \xi^a\partial_a$ .

The integrals of these equations over a compact two-surface  $N_2$  become, after a suitable normalization,

$$\frac{\partial}{\partial u}U(u, v) = \frac{1}{16\pi} \oint d^2y \left( \pi^{ab}D_+\rho_{ab} + \pi_{\sigma}D_+\sigma - hD_+\pi_h \right), \quad (29)$$

$$\frac{\partial}{\partial u}P(u, v) = \frac{1}{16\pi} \oint d^2y H, \quad (30)$$

$$\frac{\partial}{\partial u}L(u, v; \xi) = \frac{1}{16\pi} \oint d^2y \left( \pi^{ab}\mathcal{L}_{\xi}\rho_{ab} + \pi_{\sigma}\mathcal{L}_{\xi}\sigma - h\mathcal{L}_{\xi}\pi_h - A_+^a\mathcal{L}_{\xi}\pi_a \right), \quad (31)$$

where  $U(u, v)$ ,  $P(u, v)$ , and  $L(u, v; \xi)$  are defined as

$$U(u, v) := \frac{1}{16\pi} \oint d^2y \left( h \pi_h + 2e^\sigma D_+ \sigma \right) + U^0, \quad (32)$$

$$P(u, v) := \frac{1}{16\pi} \oint d^2y (\pi_h) + P^0, \quad (33)$$

$$L(u, v; \xi) := \frac{1}{16\pi} \oint d^2y (\xi^a \pi_a) + L^0. \quad (34)$$

Here  $H$  is the Hamiltonian function given by (22), and  $U^0$ ,  $P^0$ , and  $L^0$  are undetermined subtraction terms, which however must be  $u$ -independent,

$$\frac{\partial U^0}{\partial u} = \frac{\partial P^0}{\partial u} = \frac{\partial L^0}{\partial u} = 0, \quad (35)$$

in order to satisfy the equations (29), (30), and (31), respectively.

One can write the r.h.s. of the equation (29) in a more symmetric and suggestive form as follows. Let us contract the equation (26) with  $A_+^a$  and integrate the resulting equation over  $N_2$  to obtain the following equation,

$$\oint d^2y \left( A_+^a \partial_+ \pi_a \right) = \oint d^2y \left( \pi^{ab} \mathcal{L}_{A_+} \rho_{ab} + \pi_\sigma \mathcal{L}_{A_+} \sigma - h \mathcal{L}_{A_+} \pi_h \right). \quad (36)$$

If we use the definitions of  $\text{diff}N_2$ -covariant derivatives  $D_\pm$  and the equation (36), then the equation (29) can be written as

$$\frac{\partial}{\partial u} U(u, v) = \frac{1}{16\pi} \oint d^2y \left( \pi^{ab} \partial_+ \rho_{ab} + \pi_\sigma \partial_+ \sigma - h \partial_+ \pi_h - A_+^a \partial_+ \pi_a \right). \quad (37)$$

Notice that the r.h.s. of the conservation equations (31) and (37) match exactly, if we interchange the derivatives

$$\mathcal{L}_\xi \longleftrightarrow \partial_+ \quad (38)$$

in the two-surface integrals.

In a region of a spacetime where  $\partial/\partial u$  is timelike, these quasi-local conservation equations relate the instantaneous rates of changes of two-surface integrals at a given  $u$ -time to the associated net flux integrals, and they form a “complete” set of quasi-local conservation equations since they follow directly from the four Einstein’s “constraint” equations. Let us remark that, unlike the Tamburino-Winicour’s quasi-local conservation equations [7] which are “weak” conservation equations since they assumed in the derivation the Ricci flat conditions (i.e. the full Einstein’s equations), our quasi-local conservation equations are “strong” conservation equations since we used the four Einstein’s “constraint” equations only.

### III. QUASI-LOCAL ENERGY AND ENERGY-FLUX INTEGRAL

The equations (29), (30), and (31) are in fact quasi-local energy, momentum, and angular momentum conservation equations, respectively [7,16]. In this section we shall focus on the quasi-local energy conservation equation (29), and defer discussions of other conservation equations to the section IV. In order to define a quasi-local energy associated with a given two-surface  $N_2$ , we have to introduce a subtraction term  $U^0$  referring to that region only. In general the subtraction terms for a quasi-local energy are not unique, and the “right” subtraction term may not even exist at all in a generic situation. One natural criterion is that the subtraction term must be chosen such that the quasi-local energy reproduces “standard” values in limiting cases. One possible candidate is

$$U^0 = \sqrt{\frac{\mathcal{A}}{16\pi}}, \quad (39)$$

where  $\mathcal{A}$  is the area of  $N_2$ . However, this subtraction term introduces a restriction on the admissible two-surfaces  $N_2$  or, equivalently, on the vector field  $\partial/\partial u$ , due to the condition

$$\frac{\partial \mathcal{A}}{\partial u} = 0 \quad (40)$$

that follows from (35). But this is just the condition that the Bondi time function satisfies at the null infinity. Therefore the time function  $u$  satisfying the condition (40) may be regarded as a quasi-local, finite distance analog of the standard Bondi time function. The condition (40) can be interpreted as follows: In order to evaluate and compare quasi-local energies of a given two-surface  $N_2$  at two different times in a physically meaningful way, there must exist certain requirements on  $N_2$ , and the minimum requirement is that the area of the two-surface  $N_2$  is independent of  $u$ .

The quasi-local energy integral for a two surface with the area  $\mathcal{A}$  is then given by

$$U(u, v) = \frac{1}{16\pi} \oint d^2y \left( h \pi_h + 2e^\sigma D_+ \sigma \right) + \sqrt{\frac{\mathcal{A}}{16\pi}}, \quad (41)$$

which in general is not positive-definite, and does not possess monotonicity either. The lack of these properties seems to be related to the fact that gravitational binding energy is always negative. However, as we shall see shortly, the quasi-local energy function at the null infinity reduces to the Bondi mass, which is positive-definite and decreases monotonically as the time  $u$  increases.

### III-1. The Bondi mass-loss formula

In asymptotically flat spacetimes, the metric becomes

$$ds^2 \longrightarrow -2dudv - \left(1 - \frac{2m}{v}\right) du^2 + v^2 d\Omega^2 \quad (42)$$

as  $v \rightarrow \infty$ . Thus  $\partial/\partial u$  is asymptotic to the timelike Killing vector field. In general, the energy-flux across a two-surface is given by the energy-momentum tensor  $T_{0i}$ , which is of the form

$$T_{0i} \sim \pi_\phi \partial_i \phi. \quad (43)$$

The integrand of the r.h.s. of (37) is of this form, and we therefore expect that it represents the energy-flux carried by gravitational radiation crossing  $N_2$ . Then the l.h.s. of (37) should be the instantaneous rate of change in the gravitational energy of the region enclosed by  $N_2$ . The energy-flux integral in general does not have a definite sign, since it includes the energy-flux carried by the in-coming as well as the out-going gravitational radiation. But in the asymptotically flat region, the energy-flux integral turns out to be negative-definite, representing the physical situation that there is no in-coming flux coming from the infinity.

Let us now show that the equation (29) reduces to the Bondi mass-loss formula [15] in the asymptotic region of asymptotically flat spacetimes. The asymptotic fall-off rates of the metric coefficients in the asymptotic Bondi coordinates are given by [17,18]

$$e^\sigma = v^2 (\sin\vartheta) \left\{ 1 + O\left(\frac{1}{v^2}\right) \right\}, \quad (44)$$

$$\rho_{\vartheta\vartheta} = \left( \frac{1}{\sin\vartheta} \right) \left\{ 1 + \frac{\alpha}{v} + O\left(\frac{1}{v^2}\right) \right\}, \quad (45)$$

$$\rho_{\varphi\varphi} = (\sin\vartheta) \left\{ 1 + \frac{\beta}{v} + O\left(\frac{1}{v^2}\right) \right\}, \quad (46)$$

$$\rho_{\vartheta\varphi} = \frac{\gamma}{v} + O\left(\frac{1}{v^2}\right), \quad (47)$$

$$2h = 1 - \frac{2m}{v} + O\left(\frac{1}{v^2}\right), \quad (48)$$

$$A_+^a = O\left(\frac{1}{v}\right), \quad (49)$$

$$A_-^a = O\left(\frac{1}{v^2}\right), \quad (50)$$

where the expansion coefficients  $\alpha, \beta, \gamma$ , and  $m$  are functions of  $(u, \vartheta, \varphi)$ . Then the total energy at the null infinity coincides with the Bondi energy  $U_B(u)$ ,

$$\lim_{v \rightarrow \infty} U(u, v) = \frac{1}{4\pi} \oint_{S_2} d\Omega \, m(u, \vartheta, \varphi) = U_B(u), \quad (51)$$

where  $m(u, \vartheta, \varphi)$  is the mass aspect of the asymptotically flat radiating spacetime. One can easily show that the equation (29) becomes

$$\frac{d}{du}U_B(u) = -\frac{1}{32\pi} \oint d\Omega v^2 \rho^{ab} \rho^{cd} (\partial_+ \rho_{ac})(\partial_+ \rho_{bd}) \leq 0, \quad (52)$$

which is just the Bondi mass-loss formula. Notice that the negative-definite energy-flux is a bilinear of the traceless current  $j^a_b$  defined as

$$j^a_b := \rho^{ac} \partial_+ \rho_{bc} \quad (j^a_a = 0), \quad (53)$$

representing the shear degrees of freedom of gravitational radiation.

### III-2. The Misner-Sharp energy

Let us now consider a spherical ball of radius  $v$  filled with a perfect fluid with energy density  $\rho(v)$ . The Einstein's equations are modified by the presence of the fluid, and the solution is given by

$$ds^2 = -2dudv - 2h(v)du^2 + v^2 d\Omega^2, \quad (54)$$

where

$$2h(v) = 1 - \frac{2m(v)}{v}, \quad (55)$$

$$m(v) = 4\pi \int_0^v dv' v'^2 \rho(v'). \quad (56)$$

If we choose the subtraction term such that

$$U^0(v) = \sqrt{\frac{\mathcal{A}}{16\pi}} = \frac{v}{2}, \quad (57)$$

then  $U(v)$  becomes the Misner-Sharp energy  $m(v)$  [19,20]

$$U(v) = m(v). \quad (58)$$

### III-3. Black holes

One might be also interested in applying this formalism to black holes, and try to obtain quasi-local energy of the event horizon and quasi-local energy-flux incident on the horizon. For this problem, it is appropriate to choose a coordinate system adapted to the *in*-going null geodesics. In a spacetime where the metric is given by

$$ds^2 = +2dudv - 2hdu^2 + e^\sigma \rho_{ab} (dy^a + A_+^a du + A_-^a dv) (dy^b + A_+^b du + A_-^b dv), \quad (59)$$

the vector field

$$\frac{\partial}{\partial v} - A_-^a \frac{\partial}{\partial y^a} \quad (60)$$

is an in-going null vector field generating the null hypersurface  $u = \text{constant}$ , and the vector field

$$\frac{\partial}{\partial u} - A_+^a \frac{\partial}{\partial y^a} \quad (61)$$

whose norm is  $-2h$  is either timelike, null, or spacelike, depending on the sign of  $2h$ . Thus, in a region of a spacetime where  $2h = 0$ , it becomes an out-going null vector field. Therefore, on the event horizon  $H$  generated by the out-going null vector fields, we must have  $2h = 0$ . If we repeat the previous analysis in the in-going null coordinate system (59), we obtain another set of quasi-local conservation equations. On the event horizon  $H$ , the quasi-local energy conservation equation becomes

$$\frac{\partial U_H}{\partial u} = \frac{1}{16\pi} \oint_H d^2y \left\{ \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_+ \rho_{bd}) - e^\sigma (D_+ \sigma)^2 - 2e^\sigma \kappa D_+ \sigma \right\}, \quad (62)$$

$$U_H := -\frac{1}{8\pi} \oint_H d^2y (e^\sigma D_+ \sigma) + U_H^0, \quad (63)$$

where  $\kappa := D_- h|_H$  is the *surface gravity* on  $H$ . This equation is identical to the quasi-local energy conservation equation on the *stretched* horizon, which was studied in detail in [21]. Notice that when the subtraction term  $U_H^0$  is chosen zero, the quasi-local energy  $U_H$  is non-positive since the area of the event horizon always increases,

$$D_+ \sigma|_H \geq 0. \quad (64)$$

However, when the black hole no longer expands so that  $D_+ \sigma|_H = 0$ , then  $U_H$  becomes zero. For instance, for a Schwarzschild or Kerr black hole, we have  $U_H = 0$  [21–24]. This counter-intuitive aspect is a manifestation of the well-known teleological nature of the event horizon. That is, when the event horizon  $H$  evolves, its quasi-local energy must be negative so as to cancel out the positive in-flux of energy carried by subsequently in-falling matter or gravitational radiation, leaving  $U_H = 0$  when the black hole reaches the final stationary state. Details of this derivation and discussions of the remaining quasi-local conservation equations on the event horizon will be presented elsewhere [16].

#### IV. DISCUSSIONS

In summary we derived a set of four quasi-local conservation equations from the Einstein’s “constraint” equations. In particular, we showed that one of the quasi-local conservation equations reproduces the Bondi mass-loss relation in the asymptotic region of asymptotically flat spacetimes, and that the quasi-local energy coincides with the Misner-Sharp energy for spherically symmetric fluids in a finite region. We also applied the quasi-local energy conservation equation to the horizon of a general dynamic black hole, and found that it reduces to the quasi-local conservation equation of Thorne *et al.* It must be stressed that our quasi-local conservation equations are exact and unique, in that they were obtained directly from the Einstein’s “constraint” equations through the first-order canonical formalism.

It seems appropriate to mention that the equation (30) has a similar structure to the integrated Navier-Stokes equation for a viscous fluid [25],

$$\frac{\partial P_i}{\partial u} = - \oint dS^k \left( p \delta_{ik} + \rho v_i v_k - \sigma'_{ik} \right), \quad (65)$$

where  $P_i$  and  $\sigma'_{ik}$  are the total momentum and the viscous term,

$$P_i = \int dV (\rho v_i), \quad (66)$$

$$\sigma'_{ik} = \eta \left( \frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x^l} \right) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x^l}, \quad (67)$$

and  $\eta$  and  $\zeta$  are the coefficients of shear and bulk viscosity, respectively. This equation tells us that the rate of the net momentum change of a fluid within a given volume is determined by the net momentum-flux across the two-surface enclosing the volume. Notice that the Hamiltonian function  $H$  in (22) is at most quadratic in the conjugate momenta  $\pi_I$ , and assumes a form of momentum-flux of a viscous fluid. From this point of view, terms quadratic in  $\pi_I$  are responsible for direct momentum transfer, terms linear in  $\pi_I$  may be regarded as viscosity terms, and terms independent of  $\pi_I$  as pressure terms. This observation allows us to interpret the Hamiltonian function  $H$  as the gravitational momentum-flux and the two-surface integral

$$\frac{1}{16\pi} \oint d^2 y (\pi_h) \quad (68)$$

as the quasi-local gravitational momentum associated with  $N_2$ .

The equation (31) is a quasi-local angular momentum conservation equation, since the r.h.s. assumes the canonical form

$$\pi_I \mathcal{L}_\xi q^I, \quad (69)$$

representing the angular momentum-flux associated with the vector field  $\xi = \xi^a \partial_a$ . The proposed quasi-local momentum and angular momentum conservation equations will be analyzed in detail in a separate paper [16].

A final remark concerns with quantum gravity. By replacing the conjugate momenta  $\pi_I$  as

$$\pi_I \rightarrow -i \frac{\delta}{\delta q^I} \quad (70)$$

in (32), (33), and (34), one can obtain a set of functional Schrödinger equations. For instance, from the equation (32), one obtains the following equation,

$$\frac{1}{16\pi} \oint d^2y \left\{ -i\hbar \frac{\delta\Psi}{\delta h} + 2e^\sigma (D_+\sigma)\Psi \right\} = i \frac{\partial\Psi}{\partial u}, \quad (71)$$

where the subtraction term  $U^0$  was chosen zero. These functional Schrödinger equations seem worth exploring in situations where quantum gravity effects are expected to be dominant, for example, near black holes.

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